Note on Dynamic Game with Incomplete Information

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1 Solution Concept 6: Perfect Bayesian Equilibrium and Sequential Equilibrium

Definition 1.1 (Harsanyi Transformation)

- *Transform the game of incomplete information into a game of imperfect information.*
- *Introduce a prior move by Nature that determines Player 1's type (i.e., its cost).*
  - *Player 1 observes Nature's move but Player 2 can't*
  - *But Player 2 knows the probability of Nature's move*
- *Player 2's incomplete information about player 1's type becomes Player 2's imperfect information about Nature's move.*

Definition 1.2 (Assessment)

An assessment (σ, μ) in an extensive game consists of a behavioral strategy profile and a belief system, where beliefs μ at a given information set is a probability distribution on the information set.

Definition 1.3 (Sequential Rationality (Imperfect Information))

A player is sequentially rational iff, at each of his information sets, he maximizes his expected payoff given his beliefs.

Definition 1.4 (Weak Consistency)

Given any strategy profile s and any information set I on the path of play of s, a player's beliefs at I is weakly consistent with s iff the beliefs are derived using the Bayes' rule and s.

Note on Example *In this example, assume that the first and second information sets are on the path of play, and the third is off the path of play. Thus weak consistency requires that*

$$x = \frac{pq}{pq + (1-p)r} \quad \text{and}$$
$$y = \frac{p(1-q)}{p(1-q) + (1-p)(1-r)}$$

and it does not put any restriction on z.

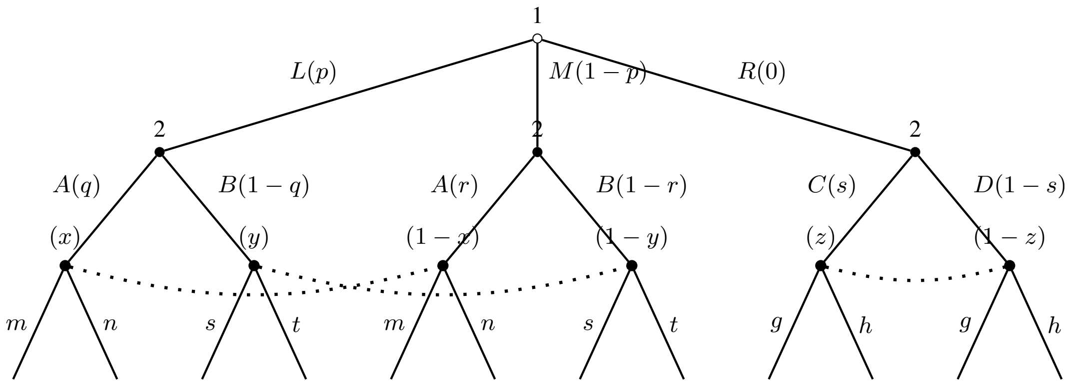


Figure 1: Example of weak consistency

Definition 1.5 (wPBE)

An assessment (σ, μ) in an extensive game is a weak perfect Bayesian equilibrium if it satisfies both sequential rationality and weak consistency.

Note on wPBE vs. NE *wPBE is also referred to as weak sequential equilibrium. A wPBE is a NE, but not every NE is a wPBE. Note that (σ, μ) is a NE if sequential rationality is satisfied on information sets on the path of play and beliefs are weakly consistent. However, (σ, μ) is a wPBE requires sequential rationality on all information sets.*

Note on Weak *The "weak" in wPBE is because of the weak consistency, we have no restrictions on beliefs at information sets that are off the path of play.*

Note on How to find wPBE *The beliefs are consistent with the strategies, which are optimal given the beliefs. Due to this circularity, wPBE cannot be determined by backward induction. To find all wPBEs, we first find all NEs, and then for each NE strategy profile σ, check whether there is a system of belief μ such that (σ, μ) satisfies both sequential rationality and weak consistency.*

Note on wPBE vs. SPE *A SPE may not be a wPBE, and a wPBE need not be a SPE.*

For example, the strategy profile of a SPE need not be a wPBE. Here (O,F) is a NE and SPE, but not a wPBE. Li duozhe p25.

For example, in Figure 2, the strategy profile of a wPBE need not be a SPE. Here the only SPNE is ((In,In-2),A), while there are xx wPBE: .

Definition 1.6 (PBE)

A Perfect Bayesian Equilibrium is a WPBE that induces a WPBE in every subgame.

Note on *PBE still does not place much restrictions on out-of-equilibrium beliefs.*

Definition 1.7 (Consistency)

An assessment (σ, μ) is consistent if there is a sequence (σ^k, μ^k) of assessment s.t.

(i) Each σ^k is completely mixed.

(ii) Each μ^k is derived from σ^k using Bayes' rule.

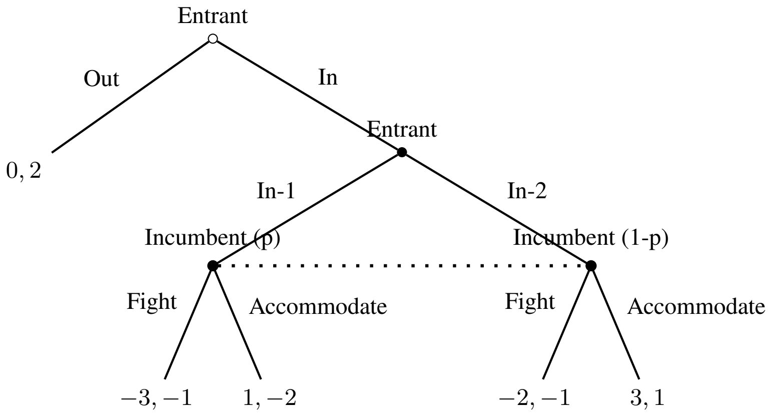


Figure 2: Example of wPBE (no SPE)

(iii) $(\sigma^k, \mu^k) \to (\sigma, \mu)$.

Note on Weak consistency vs. Consistency *Weak consistency has no restriction on the beliefs at information sets that are off the path of play. And consistency provides reasonable restrictions about the beliefs at the information sets which are off the path of play.*

Note on Interpretation *(i) means that players imaging how they make mistakes and allocate positive probability to all actions. For example, player 1 plans to choose action 1, but action 2 and 3 may be chosen by mistake. Based on (i), (ii) show that we are able to compute the probability for information sets out of the path of play. (iii) further require the strategy profile and the belief should both converge to wPBE (i.e. the chance of making mistakes becomes arbitrary small), otherwise this wPBE is not a SE.*

Definition 1.8 (SE)

An assessment (σ, μ) is a sequential equilibrium if it satisfies both sequential rationality and consistency.

Note on SE vs. SPE vs. wPBE *A SE is both a SPE and a wPBE, and SE is nearly to PBE. For example,*

Note on How to find SE *First we find all wPBE, and check whether they satisfy consitency.*

Example 1*: we first find all NEs in this game. The set of NE is ((Out,In-1), Fight), ((Out,In-2), Fight) and ((In,In-2), Accommodate).*

	Fight	Accommodate
$(\text{Out}, \text{In}-1)$	$\underline{0}, \underline{2}$	$0, 2$
$(\text{Out}, \text{In}-2)$	$\underline{0}, \underline{2}$	$0, 2$
$(\text{In}, \text{In}-1)$	$-3, -1$	$1, -2$
$(\text{In}, \text{In}-2)$	$-2, -1$	$\underline{3}, \underline{1}$

Note that ((Out,In-2), Fight) requires $p > \frac{2}{3}$ to be a wPBE, that is, the expected utility of *choosing fight is larger than the expected utility of choosing accommodate:*

$$(-1)p + (-1)(1-p) > (-2)p + 1(1-p).$$

However, ((Out,In-2), Fight) with $p > \frac{2}{3}$ *does not satisfy consistency and is not a SE. To see this, let* $\sigma_1^k(In) = \varepsilon_k$ *and* $\sigma_2(In-1) = \delta_k$, *i.e., the entrant chooses In with probability* ε_k, *and chooses In-1 with probability* δ_k. *Then according to Bayes' rule,* $p = \frac{\sigma_2^k(In-1)}{\sigma_2^k(In-1)+\sigma_2^k(In-2)} = \frac{\delta_k\varepsilon_k}{\delta_k\varepsilon_k+(1-\delta_k)\varepsilon_k} = \delta_k$, *and when* $\sigma^k \to$ *((Out,In-2), Fight), we have* $p = \delta_k \to 0$, *which contradicts* $p > \frac{2}{3}$.

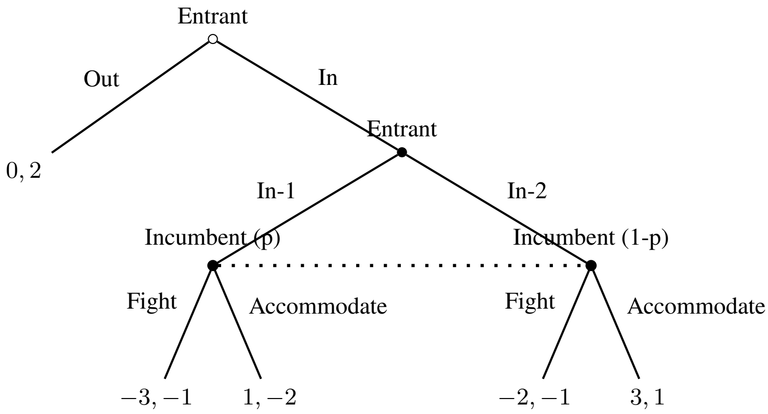


Figure 3: Example of finding SE

Example 2, the set of NE is (b,d,e) and (b,c,f).

	c	d
a	$1, 2, 0$	$-1, 1, 0$
b	$2, 1, 3$	$\underline{0}, \underline{3}, \underline{2}$

Figure 4: e

	c	d
a	$1, 2, 0$	$0, 3, 1$
b	$\underline{2}, \underline{1}, \underline{3}$	$1, -1, 0$

Figure 5: f

And the set of wPBE is (b,d,e | $\alpha = \beta = 0$*) and (b,c,f |* $\alpha = 0, \beta \geq \frac{2}{3}$*). However, (b,c,f |* $\alpha = 0, \beta \geq \frac{2}{3}$*) is not a SE because consistency is not satisfied. Suppose player 1 makes a mistake and chooses a with probability* ε_k *and player 2 chooses d with probability* δ_k. *Note that both nodes for player 3 are chosen with infinitely small probability, but the right node is more possible. Thus it should be* $\beta \to 0$.

Note on Invariancy *SE is not invariant, i.e., insensitive to inessential transformation (preserving the reduced strategic form) on the game tree. For example, Li duozhe p29.*

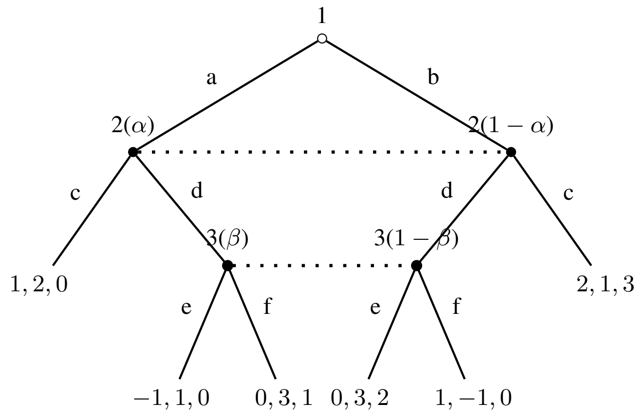


Figure 6: Example of finding SE 2

2 Signaling Game

Theorem 2.1 (Intuitive criterion)

If, under some ongoing equilibrium, a non-equilibrium signal is received which a equilibrium-dominated for some types but not others, then beliefs cannot place positive probability weight on the former set of types.

Note on *Intuitive criterion shows that no poooling equilibrium can survive intuitive criterion.*

The following papers show some interesting insights about signaling.

- Akerlof (1970): it shows that a market may function badly if the informed party has no way to signal the quality of the good it is selling.
- Spence (1973): the signal that is sent by the informed party has a cost that depends on its type so that higher types are more likely to send higher signals. This signal may then help the uninformed party to discriminate among the different types.
- Crawford-Sobel (1982): even if the signal is purely extrinsic (if it has no cost for the informed party) and thus constitutes cheap talk, both parties may still coordinate on equilibria that reveal some information.
- There are multiple equilibria in Spence's and Crawford-Sobel's models, however, equilibria in Spence's can be refined by PBE, while the latter cannot.

2.1 The market for lemons (Akerlof, 1970)

Suppose there are two types of car in the market: q proportion of "plums" (value g for seller and $G > g$ for buyer) and $1-q$ proportion of "lemons" (value l for seller and $L > l$ for buyer). Here we also assumes that $G > L, g > l$. Suppose that only seller knows the type.

Pooling equilibrium. Case 1 $L \geq g$: The seller sell all cars at $qG + (1-q)L \geq g$.

Separating equilibrium. Case 2 $L < g$: Since the seller sells "plums" at a price higher than g, thus, the seller knows the car are "lemons" if $p \leq g$. When there are two types of cars in the market, the buyer must consider that a car is worth $qG + (1-q)L < g$ and will not purchase them. Thus, the only equilibrium is that the seller sells "lemons" at $p = L < g$ and only lemons are sold.

2.2 Signal with cost (Spence, 1973)

Definition 2.1 (Spence-Mirrlees single-crossing property)

Employee side:

- Private information in productivity $\theta \in \{\theta_1, \theta_2\}$.
- Earn utility $u(w) - C(e, \theta)$ if studies for e years and is hired at wage w.
- Productivity does not depend on education, but more costly if he is by nature not productive. This means they can use diploma (public information) to signal their type (private information).

$$u' > 0, u'' < 0, \frac{\partial C}{\partial e} > 0, \frac{\partial C}{\partial \theta} > 0, \frac{\partial^2 \theta}{\partial e^2} > 0, \frac{\partial^2 C}{\partial e \partial \theta} < 0$$

Employer side:

- Provide wage $w(e) = \mu(e)\theta_1 + (1 - \mu(e))\theta_2$ if employers think that the candidate is θ_1 with probability $\mu(e)$.
- μ_0: the priori of employers on the worker's productivity.

The perfect Bayesian equilibria should consists of a vector of strategies (e_1^*, e_2^*, w^*) and a system of beliefs μ^* as follows:

$$\forall i = 1, 2 \quad e_i^* \in \arg\max_e (u(w^*(e)) - C(e, \theta_i)) \tag{1}$$

$$w^*(e) = \mu^*(e)\theta_1 + (1 - \mu^*(e))\theta_2 \tag{2}$$

$$For \quad e_1^* \neq e_2^* \qquad if \quad e = e_1^*, then \quad \mu^*(e) = 1 \tag{3}$$

$$if \quad e = e_2^*, then \quad \mu^*(e) = 0 \tag{4}$$

$$For \quad e_1^* = e_2^* \qquad if \quad e = e_1^* = e_2^*, then \quad \mu^*(e) = \mu_0 \tag{5}$$

Separating equilibrium. Low type chooses e_1^*, and high type chooses $e_2^* > e_1^*$. Low type get wages which equal to θ_1, thus, it is useless for this type to invest in study, i.e., $e_1^* = 0$. Hype type get wages θ_2. To ensure the existence of separating equilibrium, there is no incentive for low type to deviate, i.e., $u(\theta_1) - C(0, \theta_1) \geq u(\theta_2) - C(e_2^*, \theta_1)$, this characterizes a lower bound $\underline{e}$ for e_2^*; there is no incentive for high type to deviate too, i.e., $u(\theta_2) - C(e_2^*, \theta_2) \geq u(\theta_1) - C(0, \theta_2)$, this characterizes a upper bound $\overline{e}$ for e_2^*.

Pooling equilibrium. Both types chooses e^*, thus, the employer provides the same wage $\mu_0\theta_1 + (1 - \mu_0)\theta_2$ to all employee.

There is a threshold between separating and pooling equilibrium, in which low type gets the same in separating and pooling equilibrium, i.e., $\mu(\mu_0\theta_1 + (1 - \mu_0)\theta_2) - C(\overline{e}, \theta_1) = u(\theta_1) - C(0, \theta_1)$.

Hybrid equilibrium. Suppose there are p proportion of low type, and $1-p$ proportion of high type. Then in a hybrid equilibrium, low type chooses $e=0$, while high type chooses $e=0$ with prob q and e with prob $1-q$. In employer's opinion, the posterior for $e=0$ to be high type is $\frac{qp}{qp+(1-p)}$, thus, empolyer offers $\frac{qp}{qp+(1-p)}\theta_1+\frac{1-p}{qp+(1-p)}\theta_2$. The condition for high type's indifference between $e=0$ and e is

$$\frac{qp}{qp+(1-p)}\theta_1+\frac{1-p}{qp+(1-p)}\theta_2=w(e).$$

2.3 Signal without cost (Cheap Talk) (Crawford and Sobel, 1982)

There are N villagers with private costs $c_i\sim[0,1+\varepsilon]$ for hunting. Suppose all of them chooses hunting, they get 1, respectively. However, if anyone does not opt for hunting, everyone gets 0. Suppose every villagers chooses hunting with prob π, then the expected payoff of choosing hunting is π^{N-1}. And a villager chooses hunting only when c_i is smaller than π^{N-1}, i.e., π is the probability where c_i is smaller than π^{N-1}. In equilibrium, $\pi=c=0$ and no one opts for hunting.

$$\pi=\frac{c}{1+\varepsilon}=\frac{\pi^{N-1}}{1+\varepsilon}$$

Now we change this game to a sequence game, in which villagers say yes or no in the first period, and opts for hunting in the second period. Thus, if all villagers say yes, then they go for hunting together; otherwise, they all stay at home. However, there still exists babbling equilibrium in simultaneous game.

2.4 Limit Pricing

Consider an entry game with an incumbent monopolist (Firm 1) and an entrant (Firm 2) who analyzes whether or not to enter the market. The incumbent's marginal costs are either high or low, i.e., $c_1^H>c_1^L>0$. Let us consider a two-stage game here

1. In the first stage, the incumbent has monopoly power and selects an output level q.
2. In the second stage, a potential entrant decides whether or not to enter. If entry occurs, Cournot competition with x_1 and x_2, otherwise, firm 1 monopolizes the market.

Complete Information

We can apply backward induction to find the SPE. **Second period, No entry.** The incumbent chooses x_1 to maximizes $\bar{M}_1^K$, and $x_1^{K,m}$ is the profit-maximizing output, where $K=\{H,L\}$.

$$\bar{M}_1^K\equiv\max_{x_1}p\left(x_1\right)x_1-c_1^Kx_1$$

Second period, Entry. Both firms do Cournot competition, here $c_2=c_1^H$ represents the entrant's marginal cost (can be relaxed), and F denotes the fixed entry cost. To make the entry decision interesting, assume that entry occurs only when the incumbent's costs are low, i.e.,

$D_2^L < 0 < D_2^H$ for all q.

$$D_1^K \equiv \max_{x_1} p(x_1 + x_2)x_1 - c_1^K x_1 \quad \text{and} \quad D_2^K \equiv \max_{x_2} p(x_1 + x_2)x_2 - c_2 x_2 - F$$

First period. With c_L, entry does not occur, the incumbent chooses q to maximizes profits, here $\delta \in (0,1)$ denotes the discount factor, let $q^{L,Info}$ denotes the optimal solution.

$$\max_q p(q)q - c_1^L q + \delta \bar{M}_1^L$$

When the incumbent's cost is high, the incumbent chooses q to maximizes profits, and let $q^{H,Info}$ denotes the optimal solution.

$$\max_q p(q)q - c_1^H q + \delta D_1^H.$$

Importantly, under complete information, the high-cost incumbent cannot deter entry, and the low-cost incumbent doesn't need to deviate from $q^{L,Info}$ to deter entry.

Incomplete Information

The game is redesigned as follows.

1. The incumbent privately observes the realization of marginal costs, c_H with $p \in (0,1)$ and c_L with $1-p$. The incumbent chooses first-period output level q.
2. Observing q, the entrant forms beliefs $\mu\left(c_1^K \mid q\right)$ about the incumbent's marginal costs. Given these posterior beliefs, the entrant decides whether or not to enter the market.
3. With entry, they do Cournot competition, otherwise, the incumbent monopolizes the market.

Separating equilibrium. Assume this equilibrium is that the incumbent selects q^H with c_H and q^L with c_L. Entrant's equilibrium beliefs are $\mu\left(c_1^H \mid q^H\right) = 1$ and $\mu\left(c_1^H \mid q^L\right) = 0$, for simplicity we assume off-the-equilibrium beliefs are $\mu\left(c_1^H \mid q\right) = 1$ for all $q \neq q^H \neq q^L$, and the entrant enters only when it infers a high type. Now we can analyze the conditions for the existence of separating equilibrium.

High-cost incumbent. The incumbent should have not incentive to deviate to q^L, that is

$$M_1^H\left(q^{H,\text{Info}}\right) + \delta D_1^H = \max_q M_1^H(q) + \delta D_1^H \geq M_1^H\left(q^L\right) + \delta \bar{M}_1^H \tag{C1}$$

Low-cost incumbent. The incumbent should have not incentive to deviate to q^H, that is

$$M_1^L\left(q^L\right) + \delta \bar{M}_1^L \geq \max_q M_1^L(q) + \delta D_1^L = M_1^L\left(q^{L,\text{Info}}\right) + \delta D_1^L \tag{C2}$$

By solving these quadratic constraints about q^L, we can derive a feasible region $q^L \in [q^A, q^B]$.

Proposition 2.1 (Separating PBEs)

A separating strategy profile can be sustained as a Perfect Bayesian Equilibria in the signaling game where:

1. *In the first period, the high-cost incumbent selects $q^{H,Info}$ and the low-cost chooses $q^L \in [q^A, q^B]$, where q^A solves condition $C1$ with equality, and $q^A > q^{L,\text{Info}}$;*

wheras q^B solves condition $C2$ with equality.

2. *The entrant enters only after observing $q^{H,Info}$, given equilibrium beliefs $\mu\left(c_1^H \mid q^{H,\text{Info}}\right) = 1$ and $\mu\left(c_1^H \mid q^L\right) = 0$ after observing any $q^L \in \left[q^A, q^B\right]$. For every off-the-equilibrium output level $q \neq q^{H,\textit{Info}} \neq q^L$, entrant's beliefs are $\mu\left(c_1^H \mid q\right) = 1$; and*
3. *In the second period of the game, the incumbent selects an output $x_1^{K,m}$ if entry does not occur, and every firms $i = \{1, 2\}$ chooses $x_i^{K,d}$ if entry occurs.*

Proposition 2.2 (Separating PBEs survives Intuitive Criterion)

Separating equilibrium. Assume both types select the same output level q, and equilibrium beliefs are $\mu\left(c_1^H \mid q\right) = p$ and $\mu\left(c_1^L \mid q\right) = 1 - p$, and for simplicity we say off-the-equilibrium beliefs are $\mu\left(c_1^H \mid q'\right) = 1$ for any $q' \neq q$.

Entrant's response. After observing q, the entrant enters iff $pD_2^H + (1-p)D_2^L \geq 0$, that is, $p \geq \frac{-D_2^L}{D_2^H - D_2^L} \equiv \bar{p}$. We hence conclude that the entrant enters if $p \geq \bar{p}$, and stays out otherwise. In particular, it must be that $p < \bar{p}$, otherwise with entry, the incumbent must deviates to $q^{K,Info}$, and since $q^{H,\text{ Info}} \neq q^{L,\text{ Info}}$, this strategy profile cannot be a pooling equilibrium.

Incumbent's IC constraints. Again, check its IC constraints. A high-cost incumbent does not deviate from q if (C1), and a low-cost incumbent does not deviate from q if (C2). By solving these quadratic constraints about q, we can derive a feasible region $q \in [q^C, q^D]$.

$$M_1^H(q) + \delta \bar{M}_1^H \geq \max_q M_1^H(q) + \delta D_1^H = M_1^H\left(q^{H,\text{ Info}}\right) + \delta D_1^H \tag{C1}$$

$$M_1^L(q) + \delta \bar{M}_1^L \geq \max_q M_1^L(q) + \delta D_1^L = M_1^L\left(q^{L,\text{Info}}\right) + \delta D_1^L \tag{C2}$$

Proposition 2.3 (Pooling PBEs)

The following strategy profiles can be sustained as pooling PBE:

1. *In the first period, both types of incumbent select the same first-period output $q \in [q^C, q^D]$, where q^C solves condition (C2) with equality, while q^D solves (C1) with equality.*
2. *The entrant does not enter after observing the equilibrium output $q \in [q^C, q^D]$, but enters after observing off-the-equilibrium output $q' \neq q$, given beliefs $\mu\left(c_1^H \mid q^{L,\textit{ Info}}\right) = p < \bar{p}$ and $\mu\left(c_1^H \mid q'\right) = 1$; and*
3. *In the second period of the game, the incumbent selects $x_1^{K,m}$ if entry does not occur, and every firm $i = \{1, 2\}$ chooses $x_i^{K,d}$ if entry occurs.*

Proposition 2.4 (Pooling PBEs survives Intuitive Criterion)

3 Sorting Game

4 Screening Game

5 Moroal Hazard

6 Solution Concept 7: Treambling-hand Perfect Equilibrium

Definition 6.1 (Trembling-Hand perfect equilibrium)

A strategy profile σ is a trembling-hand perfect equilibrium if there exists a sequence of totally mixed strategy profiles $\sigma^n \to \sigma$ such that, for all i,

$$u_i\left(\sigma_i, \sigma_{-i}^n\right) \geq u_i\left(a_i, \sigma_{-i}^n\right) \text{ for all } a_i \in A_i.$$

Note on Example *In G_1, NE (B,R) is not trembling-hand perfect. In G_2, both NE (A,A) and (C,C) are not perfect.*

	L(ε)	R($1-\varepsilon$)
T	$\underline{1}, \underline{1}$	$0, 0$
B	$0, 0$	$\underline{0}, \underline{0}$

Figure 7: G_1

	A(ε)	B(ε)	C($1-2\varepsilon$)
A	$\underline{0}, \underline{0}$	$0, 0$	$0, 0$
B	$0, 0$	$\underline{1}, \underline{1}$	$2, 0$
C	$0, 0$	$0, 2$	$\underline{2}, \underline{2}$

Figure 8: G_2

Note on Weakly dominated strategy *Trembling-hand perfection rules out the use of weakly dominated strategies. In two-player games, any NE in which neither player uses a weakly dominated strategy is trembling-hand perfect, but it is not true for games with more than two players. NE (D, L, A) is undominated, since given (R,B), D is better than U. But it is not trembling-hand perfect. To see this, say that player 2 and 3 may make mistakes in (D,L,A), then the utility for choosing U is greater than for choosing D, even though the difference is infinitely small.*

$$(1-\varepsilon_k)(1-\delta_k) + 0 + 0 + 1\varepsilon_k\delta_k < (1-\varepsilon_k)(1-\delta_k) + 0 + +\delta_k(1-\varepsilon_k) + \varepsilon_k(1-\delta_k)$$

	$L(1-\delta_k)$	$R(\delta_k)$
U	$1, 1, 1$	$1, 0, 1$
D	$1, 1, 1$	$0, 0, 1$

$A(1-\varepsilon_k)$

	$L(1-\delta_k)$	$R(\delta_k)$
U	$1, 1, 0$	$0, 0, 0$
D	$0, 1, 0$	$1, 0, 0$

$B(\varepsilon_k)$

Lemma 6.1

Every finite strategic game has a trembling-hand perfect equilibrium.

Definition 6.2 (Strictly perfect equilibrium)

A strategy profile σ is a strictly perfect equilibrium if each player's strategy is optimal against all possible (not just one) sequences of perturbations.

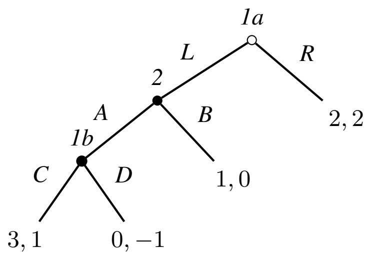
Note on Existence *Such an equilibrium might not exist. See, for example, both (T,L) and (T,R) are THBE, e.g. $(\sigma_1^k(T) = 1 - \varepsilon_k - \varepsilon_k^2, \sigma_1^k(C) = \varepsilon_k, \sigma_1^k(B) = \varepsilon_k^2)$ for (T,L) and $(\sigma_1^k(T) = 1 - \varepsilon_k - \varepsilon_k^2, \sigma_1^k(C) = \varepsilon_k^2, \sigma_1^k(B) = \varepsilon_k)$ for (T,R). But these two NEs can not be THPE for both two sequences, thus they are not strictly perfect equilibrium.*

	L	R
T	3, 2	2, 2
C	1, 1	0, 0
B	0, 0	1, 1

Definition 6.3 (Agent-strategic form (Selten, 1975))

Each information set is manned by a different "agent", and all agents of the same player have the same payoff.

Note on Origin *For an extensive game, trembling-hand perfection in its strategic form is not totally satisfactory. In the strategic form, (R,B) is trembling-hand perfect, but the unique SPE of the game is (LC,A).*



extensive form

	A	B
R	2, 2	2, 2
LC	3, 1	1, 0
LD	0, −1	1, 0

strategic form

Selten (1975) considers the agent-strategic form, and here the only THPE of the agent-strategic form is the unique SPE of the extensive form. Actually, here D is weakly dominated by C for agent 1b.

	A	*B*
R	2, 2, 2	2, 2, 2
L	3, 1, 3	1, 0, 1

Agent 1b: C

	A	B
R	2, 2, 2	2, 2, 2
L	0, −1, 0	1, 0, 1

Agent 1b: D

Definition 6.4 (Perfect equilibrium)

For an extensive game, the THPE of the agent-strategic form is referred to as perfect equilibrium.

Note on Perfect vs. Sequential *A perfect equilibrium must be sequential, but the converse is not true; for generic games the two concepts coincide. That is, perfect is stronger than sequential.*

For example, (B,D) is sequential but not perfect. Actually, we do not require σ_i^n to be BR to σ_{-i}^n in SE, we just require σ to be BR to σ_{-i} (in convergence).

	C	D
A	$1,1$	$0,0$
B	$0,0$	$0,0$

Note on THPE in agent-strategic form and strategic form *The set of THPE of the agent-strategic form of an extensive game is NOT a subset of the set of THPE of the corresponding strategic form. For example, p35.*

Note on Invariancy *Perfect equilibrium is not invariant. For example, p35.*

Theorem 6.1

In finite games, at least one perfect equilibrium exists.

7 Solution Concept 8: Proper Equilibrium

Definition 7.1 (Proper equilibrium (Myerson 1978))

An ε-proper equilibrium is a totally mixed strategy profile σ^{ε} such that, if

$$u_i\left(a_i, \sigma_{-i}^{\varepsilon}\right) < u_i\left(a_i', \sigma_{-i}^{\varepsilon}\right)$$

then $\sigma_i^{\varepsilon}\left(a_i\right) \leq \varepsilon \sigma_i^{\varepsilon}\left(a_i'\right)$. A proper equilibrium σ is any limit of ε-proper equilibria σ^{ε} as ε tends to 0.

Note on Interpretation *The basic idea is that players are less likely to make "mistakes" that are more costly.*

Note on Proper vs. perfect *A proper equilibrium must be perfect. For example, there are three NEs: (U,L), (M,C) and (D,R). T-H perfection rules out (D,R), but not (M,C), but properness rules out (M,C). Assume that player 2 chooses L,C,R with probability ε_L, $1-\varepsilon_L-\varepsilon_R$ and ε_R, then when $\sigma^k \to$ (M,C), the payoff of choosing U is greater than choosing D, then it requires $\varepsilon_D \leq \varepsilon \delta_U$. Similarly, for player 2, it requires $\varepsilon_R \leq \varepsilon \varepsilon_L$.*

	L(ε_L)	C($1-\varepsilon_L-\varepsilon_R$)	R(ε_R)
U(δ_U)	$1,1$	$0,0$	$-9,-9$
M($1-\varepsilon_U-\varepsilon_D$)	$0,0$	$0,0$	$-7,-7$
D(δ_D)	$-9,-9$	$-7,-7$	$-7,-7$

A proper equilibrium of a strategic-form game need not be a treambling-hand perfect equilibrium in the agent-strategic form of every extensive game with the given (reduced) strategic form. For example, LA is proper in the strategic form, but is not perfect in the extensive form. p39.

Note on Proper vs. backward induction outcome *Proper equilibrium yields backward induction outcome without the use of the agent-strategic form. For example, p37.*

Note on Invariancy *Proper equilibrium is invariant. More precisely, every proper equilibrium of a strategic-form game is sequential in every extensive game with the given (reduced) strategic form. For example, p38.*

Definition 7.2 (Fully reduced strategic form)

Eliminating any pure strategy that is equivalent to a mixed strategies with support excluding it.

Note on Fully invariancy *Proper equilibrium is not fully invariant.*

Theorem 7.1

Every finite strategic game has a proper equilibrium.

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